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Second-order second-degree Painlevé equations related with Painlevé I–VI equations and Fuchsian-type transformations

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One-to-one correspondence between the Painlevé I–VI equations and certain second-order second-degree equations of Painlevé type is investigated. The transformation between the Painlevé equations and second-order second-degree equations is the one involving the Fuchsian-type equation. © *1999 American Institute of Physics.* [S0022-2488(99)01507-8]

I. INTRODUCTION

Painlevé,[1] Gambier,[2] and Fuchs[3] addressed a question raised by E. Picard concerning the second-order first-degree ordinary differential equations of the form

$$v''=F(z,v,v'), \tag{1.1}$$

where F is rational in v', algebraic in v, and locally analytic in z, and have the property that all movable singularities of all solutions are poles. Movable means that the position of the singularities varies as a function of initial values. A differential equation is said to have the Painlevé property if all solutions are single valued around all movable singularities. Within the Möbius transformation, Painlevé and his school found 50 such equations. Among all these equations, 6 of them are irreducible and define classical Painlevé transcendents, PI, PII,...,PVI,[4] and the remaining 44 equations are either solvable in terms of known functions or can be transformed into one of the 6 equations. These equations maybe regarded as the nonlinear counterparts of some classical special equations. For example, PII has solution which has similar properties as Airy's functions.[5] Although the Painlevé equations were discovered from strictly mathematical considerations, they have appeared in many physical problems, and possess a rich internal structure. The properties and the solvability of the Painlevé equations have been extensively studied in the literature.[6–11]

The Riccati equation is the only example for the first-order first-degree equation which has the Painlevé property. Before the work of Painlevé and his school, Fuchs[3,4] considered the equation of the form

$$F(z,v,v')=0, \tag{1.2}$$

where F is polynomial in v and v' and locally analytic in z, such that the movable branch points are absent, that is, the generalization of the Riccati equation. Briot and Bouquet[4] considered the subcase of (1.2), that is, the first-order binomial equations of degree $m\in\mathbb{Z}_+$:

$$(v')^m+F(z,v)=0, \tag{1.3}$$

where $F(z,v)$ is a polynomial of degree at most $2m$ in v. It was found out that there are six types of equations of the form (1.3). But, all these equations are either reducible to a linear equation or solvable by means of elliptic functions.[4] Second-order binomial-type equations of degree $m\geqslant 3$,

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$$(v'')^m + F(z,v,v') = 0, \tag{1.4}$$

where F is polynomial in v and v' and locally analytic in z, were considered by Cosgrove,[12] who found out that there are nine classes. Only two of these classes can have an arbitrary degree m, and the others can have the degrees of three, four, and six. As in the case of first-order binomial-type equations, all nine classes are solvable in terms of the first, second, and fourth Painlevé transcendents, elliptic functions, or by quadratures. Chazy,[13] Garnier,[14] and Bureau[15] considered the third-order differential equations possessing the Painlevé property of the following form:

$$v''' = F(z,v,v',v''), \tag{1.5}$$

where F is assumed to be rational in v,v',v'' and locally analytic in z. But, in Ref. 15 the special form of $F(z,v,v',v'')$,

$$F(z,v,v',v'') = f_1(z,v)v'' + f_2(z,v)(v')^2 + f_3(z,v)v' + f_4(z,v), \tag{1.6}$$

where $f_k(z,v)$, $k=1,...,4$, are polynomials in v of degree k with analytic coefficients in z, was considered. In this class, no new Painlevé transcendent was discovered since, and all of them can be solved either in terms of known functions or one of the six Painlevé transcendents.

Second-order second-degree Painlevé type equations of the following form,

$$(v'')^2 = E(z,v,v')v'' + F(z,v,v'), \tag{1.7}$$

where E and F are assumed to be rational in v,v' and locally analytic in z, were subject the articles.[16,17] A special case of (1.7), given as

$$v'' = M(z,v,v') + \sqrt{N(z,v,v')}, \tag{1.8}$$

was considered in Ref. 16, where M and N are polynomials in v' of degree 2 and 4, respectively, rational in v, and locally analytic in z, and no new Painlevé transcendent was found. In Ref. 17, the special form of (1.7), $E=0$ and thus F polynomial in v and v', was considered and six distinct classes of equations denoted by SD-1,...,SD-VI, were obtained by using the α-method. Also, these classes can be solved in terms of classical Painlevé transcendents (PI,...,PVI), elliptic functions, or solutions of linear equations.

Let $v(z)$ be a solution of any of the 50 Painlevé equations, as listed by Ince,[4] each of which takes the form

$$v'' = P_2(v')^2 + P_1 v' + P_0, \tag{1.9}$$

where P_0,P_1,P_2 are functions of v, z, and a set of parameters α. The transformation, that is, Lie-point symmetry, which preserves the Painlevé property of (1.9), of the form $u(z;\hat{\alpha}) = f(v(z:\alpha),z)$ is the Möbius transformation:

$$u(z;\hat{\alpha}) = \frac{a_1(z)v + a_2(z)}{a_3(z)v + a_4(z)}, \tag{1.10}$$

where $v(z;\alpha)$ solves (1.9) with a set of parameters α and $u(z;\hat{\alpha})$ solves (1.9) with a set of parameters $\hat{\alpha}$. Lie-point symmetry can be generalized by involving $v'(z;\alpha)$, that is, the transformation of the form $u(z;\hat{\alpha}) = F(v'(z;\alpha),v(z;\alpha),z)$. The only transformation which contains v' linearly is the one involving the Riccati equation, that is,

$$u(z;\hat{\alpha}) = \frac{v' + av^2 + bv + c}{dv^2 + ev + f}, \tag{1.11}$$

where a,b,c,d,e,f are functions of z only.

In Ref. 6, the transformation of type (1.11) was used and the aim was to find a,b,c,d,e,f such that (1.11) defines a one-to-one invertible map between solutions v of (1.9) and solutions u of some second-order equations of the Painlevé type. An algorithmic method was developed to investigate the transformation properties of the Painlevé equations, and some new second-degree equations of Painlevé type related with PIII and PVI were also found. Therefore, second-degree equations are important in determining the transformation properties of the Painlevé equations.[18,6] Moreover, second-degree equations of Painlevé type appear in physics.[19–21] Furthermore, second-degree equations also appear as the first-integral of some of the third-order Painlevé-type equations.

Instead of considering the transformation of the form (1.11) one may consider the following transformation:

$$u(z;\hat{\alpha})=\frac{(v')^m+\Sigma_{j=1}^m P_j(z,v)(v')^{m-j}}{\Sigma_{j=1}^m Q_j(z,v)(v')^{m-j}}, \tag{1.12}$$

where P_j, Q_j are polynomials in v, whose coefficients are meromorphic functions of z and satisfy the Fuchs theorem[4,22] concerning the absence of the movable critical points. A second-order second-degree algebraic differential equation of the form

$$a_1(v'')^2+a_2v''v'+a_3v''v+a_4(v')^2+a_5v'v+a_6v^2=0, \tag{1.13}$$

where a_j, $j=1,2,...,6$, are meromorphic functions of z, was considered by P. Appell.[23] In Ref. 22, it was shown that Appell's condition for solvability of (1.13) is a necessary and sufficient condition for (1.13) to have its solutions free of movable branch points. Also, in Ref. 22, some analogous conditions were applied to irreducible first-order algebraic equations of the second degree, and necessary and sufficient conditions for the solutions of such equations to be free of movable branch points were obtained. A first-order algebraic differential equation of degree $n\geqslant 1$ is given as

$$a_1(z,v)(v')^n+a_2(z,v)(v')^{(n-1)}+\cdots+a_{n-1}(z,v)v'+a_n(z,v)=0, \tag{1.14}$$

where the functions $a_i(z,v)$, $i=1,...,n$, are assumed to be polynomials in v, whose coefficients are analytic functions of z. The necessary and sufficient conditions for the solutions of (1.14) to be free from movable branch points are given by the Fuchs theorem [Ref. 4 (Chap. XIII) and Ref. 22 (theorem 1.1)]. The Fuchs theorem shows that, apart from the other conditions, the irreducible form of the first-order algebraic differential equation of the second degree is

$$a_1(z)(v')^2+[a_2(z)v^2+a_3(z)v+a_4(z)]v'+[a_5(z)v^4+a_6(z)v^3+a_7(z)v^2+a_8(z)v+a_9(z)]=0, \tag{1.15}$$

where $a_i(z)$, $i=1,2,...,9$, are analytic functions of z and $a_1(z)\neq 0$. Let

$$F(v):=A_0v^4+A_1v^3+A_2v^2+A_3v+A_4, \tag{1.16}$$

where

$$\begin{gathered}A_0=4a_1a_5-a_2^2,\quad A_1=4a_1a_6-2a_2a_3,\\ A_2=4a_1a_7-2a_2a_4-a_3^2,\quad A_3=4a_1a_8-2a_3a_4,\\ A_4=4a_1a_9-a_4^2,\end{gathered} \tag{1.17}$$

It is known that when $F(v)\neq 0$, there are unique monic polynomials $F_1(v),F_2(v)$ such that

$$F(v)\equiv A(z)F_1(v)[F_2(v)]^2, \tag{1.18}$$

where $A(z)$ is an analytic function and $F_1(z)$ has no multiple roots. In Ref. 22 it was shown (theorem 6.2) that the solutions of the equation (1.15) are free of movable branch points if and only if the following conditions hold:

$$
\begin{aligned}
&(i)\ \ F_1(v) \quad \text{divides} \quad G_1(v) := (a_2v^2+a_3v+a_4)\frac{\partial F_1}{\partial v} - 2a_1\frac{\partial F_1}{\partial z},\\
&(ii)\ \ A_0=0 \quad \text{and } A_1\neq 0 \quad \text{imply}\ \ a_2=0,\\
&(iii)\ \ A_0=A_1=A_2=0 \quad \text{and } A_3\neq 0 \quad \text{imply}\ \ a_2=0.
\end{aligned}
\tag{1.19}
$$

The conditions of the Fuchs theorem are satisfied if and only if the conditions (1.19) are satisfied.

In this article, we investigate one-to-one correspondence between PI–PVI and some second-order second-degree Painlevé-type equations such that the transformation involving Eq. (1.15) is used and given by

$$
u=\frac{(v')^2+(a_2v^2+a_1v+a_0)v'+b_4v^4+b_3v^3+b_2v^2+b_1v+b_0}{(c_2v^2+c_1v+c_0)v'+d_4v^4+d_3v^3+d_2v^2+d_1v+d_0},
\tag{1.20}
$$

where a_j,b_k,c_j,d_k, $j=0,1,2$, $k=0,1,2,3,4$, are functions of z and a set of parameters α. By using the transformations of the form (1.20), second-order second-degree Painlevé-type equations which are labeled as SD-I.a, SD-I.b, SD-I.c, SD-I.d, and SD-I.e in Ref. 17, can be obtained from PVI, PIII and PV, PIV, PII, PI, respectively. In the following sections, we first present the procedure to obtain these known equations, and for each Painlevé equation we provide an example of a second-order second-degree Painlevé-type equation that has not been considered in the literature.

The procedure used to obtain second-degree Painlevé-type equations and one-to-one correspondence with PI–PVI is as follows: Given Eq. (1.9), determine a_j,b_k,c_j,d_k, $j=0,1,2,3$, $k=0,1,2,3,4$, by requiring that (1.20) defines a one-to-one map between the solution v of (1.9) and solution u of some second-degree equation of the Painlevé type. Let $A_j:=c_ju-a_j$, $B_k:=d_ku-b_k$. Then the transformation (1.20) can be written as

$$
(v')^2=(A_2v^2+A_1v+A_0)v'+B_4v^4+B_3v^3+B_2v^2+B_1v+B_0.
\tag{1.21}
$$

It should be noted that if Eq. (1.21) is reducible, that is, if there exits a nontrivial factorization, then it can be reduced to a Riccati equation. If it is not reducible, then its solutions are free of movable branch points provided that the conditions given in (1.19) are satisfied. Differentiating Eq. (1.21) and using (1.9) to replace v'' and (1.21) to replace $(v')^2$, one gets

$$
\Phi v'+\Psi=0,
\tag{1.22}
$$

where

$$
\begin{aligned}
\Phi=&(P_1-2A_2v-A_1)(A_2v^2+A_1v+A_0)+P_2(A_2v^2+A_1v+A_0)^2+2P_0-4B_4v^3-(3B_3+A_2')v^2\\
&-(2B_2+A_1')v-(B_1+A_0')+2P_2(B_4v^4+B_3v^3+B_2v^2+B_1v+B_0),\\
\Psi=&(B_4v^4+B_3v^3+B_2v^2+B_1v+B_0)[P_2(A_2v^2+A_1v+A_0)+2P_1-2A_2v-A_1]\\
&-P_0(A_2v^2+A_1v+A_0)-(B_4'v^4+B_3'v^3+B_2'v^2+B_1'v+B_0').
\end{aligned}
\tag{1.23}
$$

There are two cases to be distinguished:

(I) $\Phi=0$: Equation (1.22) becomes

$$
\Psi=0.
\tag{1.24}
$$

If the solutions of the equation (1.21) are free of movable branch points, that is, the conditions given in (1.19) are satisfied, then one obtains the Painlevé-type equation of degree $n>1$ related with PI–PVI equations. To obtain the second-degree Painlevé-type equations, one should reduce the equation (1.24) to a linear equation in v. If (1.24) is reduced to an equation which is quadratic in v, then one obtains the second-order fourth-degree Painlevé-type equations related with PI–PVI, which are not considered in this article. Hence, one can find a_j, b_k, c_j, d_k such that (1.24) reduces to a linear equation in v,

$$A(u',u,z)v+B(u',u,z)=0, \tag{1.25}$$

then, substitute $v=-B/A$ into Eq. (1.21) to determine the second-order second-degree equation of the Painlevé type for u.

(II) $\Phi\neq 0$: If Φ divides Ψ, then (1.21) can be reduced to a Riccati equation and hence its solutions are free of movable branch points. Then, one can substitute $v'=-\Psi/\Phi$ in Eq. (1.21) and obtain the following equation for v:

$$\Psi^2+(A_2v^2+A_1v+A_0)\Phi\Psi-\Phi^2(B_4v^4+B_3v^3+B_2v^2+B_1v+B_0)=0. \tag{1.26}$$

Finding a_j, b_k, c_j, and d_k such that (1.26) reduces to a quadratic equation in v,

$$A(u',u,z)v^2+B(u',u,z)v+C(u',u,z)=0. \tag{1.27}$$

Solving the equation (1.27) for v and substituting into equation (1.22) yields a second-order second-degree Painlevé-type equation for u.

It turns out that PI admits transformations discussed in cases I and II, and PII–PVI admit only transformations of case II.

Second-order second-degree Painlevé-type equations were studied mainly by Bureau and Cosgrove.[16,17] But, as mentioned before, in both articles the special form of the second-degree Painlevé-type equations was considered, and no new Painlevé transcendent was found. In Refs. 24 and 25 the transformation (1.11) was used to obtain one-to-one correspondence between PI–PVI and certain second-degree Painlevé-type equations. Some of these second-degree equations had been obtained previously, but most of them had not been considered in the literature before. In this article, we investigate the transformation of type (1.20) to obtain the one-to-one correspondence between PI–VI and the second-order second-degree Painlevé-type equations. By using the transformation of type (1.11) and the procedure described above, it is possible to obtain all of the second-degree equations given in Ref. 17 except the ones which can be solvable in terms of elliptic functions or solutions of linear equations. In addition to known equations which are related with Painlevé equations through the transformation (1.20), it is possible to obtain some new second-degree equations of the Painlevé type. Since the calculations are extremely tedious, one new second-degree Painlevé-type equation for each Painlevé equation, PI–PVI, is given. Throughout this article $'$ denotes the derivative with respect to z and $\cdot$ denotes the derivative with respect to x.

II. PAINLEVÉ I

Let $v(z)$ be a solution of PI equation,

$$v''=6v^2+z. \tag{2.1}$$

Then, for PI the equation (1.22) takes the form of

$$(\phi_3v^3+\phi_2v^2+\phi_1v+\phi_0)v'+\psi_5v^5+\psi_4v^4+\psi_3v^3+\psi_2v^2+\psi_1v+\psi_0=0, \tag{2.2}$$

where

$$\phi_3=2(A_2^2+2B_4),\quad \phi_2=A_2'+3B_3+3A_1A_2-12,$$

$$\phi_1=A_1'+2B_2+A_1^2+2A_0A_2,\quad \phi_0=A_0'+B_1+A_0A_1-2z,$$

$$\psi_5=2A_2B_4,\quad \psi_4=B_4'+A_1B_4+2A_2B_3+6A_2, \tag{2.3}$$

$$\psi_3=B_3'+A_1B_3+2A_2B_2+6A_1,\quad \psi_2=B_2'+A_1B_2+2A_2B_1+6A_0+zA_2,$$

$$\psi_1=B_1'+A_1B_1+2A_2B_0+zA_1,\quad \psi_0=B_0'+A_1B_0+zA_0.$$

Case I: $\Phi=0$: One should choose $c_j=0$, $j=0,1,2$, $d_k=0$, $k=1,2,3,4$, $b_4=\frac{1}{2}a_2^2$, $b_3=-\frac{1}{3}a_2'+a_1a_2-4$, $b_2=-\frac{1}{2}a_1'+\frac{1}{2}a_1^2+a_0a_2$, $b_1=-a_0'+a_0a_1-2z$. One can always absorb b_0 and d_0 in u by a proper Möbius transformation. Hence, without loss of generality, one can set $b_0=0$ and $d_0=2$. The only possibility to reduce the equation $\Psi=0$ to a linear equation in v is to set $\psi_5=\psi_4=\psi_3=\psi_2=0$. Therefore, one obtains $a_2=a_1=a_0=b_4=b_2=0$, $b_3=-4$, and $b_1=-2z$. Then the equation (1.20) becomes

$$2u=(v')^2-4v^3-2zv, \tag{2.4}$$

and the linear equation for v reads

$$v+u'=0. \tag{2.5}$$

Equation (2.4) with the condition (2.5) satisfies corollary 6.3 in Ref. 22, and hence its solutions are free of movable branch points. By following the procedure discussed in the Introduction, one can get the following second-order second-degree equation for $u(z)$:

$$(u'')^2=-4(u')^3-2(zu'-u). \tag{2.6}$$

Equation (2.6) was first obtained by Cosgrove[17] and labeled as SD-I.e.

Case II: $\Phi\neq 0$: As an example, let $\phi_i=0$, $i=1,2,3$, $\phi_0\neq 0$, and $\psi_l=0$, $l=2,3,4,5$. These choices imply that $A_j=0$, $j=0,1,2$, $B_4=B_2=0$, and $B_3=4$. Then, Eq. (1.26) becomes

$$(B_1'v+B_0')^2-(B_1-2z)^2(4v^3+B_1v+B_0)=0. \tag{2.7}$$

To reduce the equation (2.7) to a quadratic equation for v, one has to take $d_1\neq 0$ and, hence, without loss of generality, $b_1=0$ and $d_1=1$. Moreover, d_0 and b_0 are the solutions of the following equations:

$$d_0'(b_0'-2zd_0')=0,\quad (d_0')^2+4d_0^3+b_0=0,\quad (b_0')^2-4z^2(d_0')^2=0. \tag{2.8}$$

Here, we only consider the case $d_0'=0$; then $d_0=\mu$ and $b_0=-4\mu^3$, where μ is a constant. Therefore, the equations (1.21) and (1.22) become

$$(v')^2=4v^3+uv+\mu(u+4\mu^2) \tag{2.9}$$

and

$$v'=\frac{-u'}{(u-2z)}(v+\mu), \tag{2.10}$$

respectively, and the quadratic equation for v takes the form of

$$4(u-2z)^2v^2-[(u')^2+4\mu(u-2z)^2]v-\mu(u')^2+(u+4\mu^2)(u-2z)^2=0. \tag{2.11}$$

Let $u(z)=-2(e^x/(y-1)+6\mu^2)$ and $z=e^x-6\mu^2$. Then the equations (2.9) and (2.11) give one-to-one correspondence between solutions $v(z)$ of PI and solutions $y(x)$ of the following second-order second-degree Painlevé-type equation

$$\{4y(y-1)(\ddot{y}-\dot{y})-(\dot{y}-y+1)[(7y-4)\dot{y}+5y(y-1)]+12\mu e^{2x}y^3(y-1)^2\}^2$$
$$=(y+2)^2\{[(\dot{y}-y+1)^2+12\mu e^{2x}y^2(y-1)^2]^2+32e^{5x}y^4(y-1)^3\}. \tag{2.12}$$

III. PAINLEVÉ II

Let $v(z)$ be a solution of PII equation

$$v''=2v^3+zv+\alpha. \tag{3.1}$$

Then, for PII, the equation (1.22) takes the following form:

$$(\phi_3 v^3+\phi_2 v^2+\phi_1 v+\phi_0)v'+\psi_5 v^5+\psi_4 v^4+\psi_3 v^3+\psi_2 v^2+\psi_1 v+\psi_0=0, \tag{3.2}$$

where

$$\begin{gathered}\phi_3=4(B_4+\tfrac{1}{2}A_2^2-1),\quad \phi_2=A_2'+3B_3+3A_1A_2,\\ \phi_1=A_1'+2B_2+2A_0A_2+A_1^2-2z,\quad \phi_0=A_0'+B_1+A_0A_1-2\alpha,\\ \psi_5=2A_2(B_4+1),\quad \psi_4=B_4'+A_1B_4+2A_2B_3+2A_1,\\ \psi_3=B_3'+A_1B_3+2A_2B_2+2A_0+zA_2,\quad \psi_2=B_2'+A_1B_2+2A_2B_1+zA_1+\alpha A_2,\\ \psi_1=B_1'+A_1B_1+2A_2B_0+zA_0+\alpha A_1,\quad \psi_0=B_0'+A_1B_0+\alpha A_0.\end{gathered} \tag{3.3}$$

Here, we only consider the case $\phi_i=0$, $i=1,2,3$, $\phi_0\neq 0$, and $\psi_l=0$, $l=3,4,5$. $\psi_5=0$ implies that either $A_2=0$ or $B_4=-1$.

Case i: If $A_2=0$, then one obtains $A_1=A_0=0$, $B_4=1$, $B_3=0$, $B_2=z$ and $\phi_0=B_1-2\alpha$, $\psi_2=1$, $\psi_1=B_1'$, $\psi_0=B_0'$. With these choices, the equation (1.26) yields

$$(v^2+B_1'v+B_0')^2-(B_1-2\alpha)^2(v^4+zv^2+B_1v+B_0)=0. \tag{3.4}$$

To reduce the equation (3.4) to a quadratic equation in v, one possibility is to set the coefficients of v^4 and v^3 to zero. Then, one obtains $B_1=2\alpha+\epsilon$, where $\epsilon=\pm 1$, and by using the proper Möbius transformation, one may take $B_0=2u+\frac{1}{4}z^2$. Hence, the equations (1.21) and (1.22) become

$$(v')^2=v^4+zv^2+(2\alpha+\epsilon)v+2u+\frac{z^2}{4} \tag{3.5}$$

and

$$v'=\epsilon\left(v^2+2u'+\frac{z}{2}\right), \tag{3.6}$$

respectively. The quadratic equation in v is

$$4u'v^2-(2\alpha+\epsilon)v+4(u')^2+2(zu'-u)=0. \tag{3.7}$$

The equations (3.5) and (3.7) give one-to-one correspondence between solutions $v(z)$ of PII and solutions $u(z)$ of the equation

$$(u'')^2 = -4(u')^3 - 2u'(zu'-u) + \tfrac{1}{16}(2\alpha+\epsilon)^2. \tag{3.8}$$

The equation (3.8) was first obtained by Cosgrove[17] and labeled as SD-I.d.

Case ii: If $B_4=-1$, then one obtains $A_2=2\epsilon$, $A_1=0$, $A_0=\epsilon z$, $B_3=0$, $B_2=-z$ and $\phi_0 = B_1 - 2\alpha + \epsilon \neq 0$, $\psi_2 = 4\epsilon B_1 + 2\epsilon\alpha - 1$, $\psi_1 = B_1' + 4\epsilon B_0 + \epsilon z^2$, $\psi_0 = B_0' + \epsilon\alpha z$, where $\epsilon = \pm 1$. Then the equation (1.26) becomes

$$(\psi_2 v^2 + \psi_1 v + \psi_0)^2 + 2\epsilon\phi_0(v^2 + \tfrac{1}{2}z)(\psi_2 v^2 + \psi_1 v + \psi_0) + \phi_0^2(v^4 + zv^2 - B_1 v - B_0) = 0. \tag{3.9}$$

To reduce the equation (3.9) to a quadratic equation in v one may set the coefficients of v^4 and v^3 to zero. Thus one obtains $B_1=0$, and without loss of generality one may take $B_0 = \tfrac{1}{4}(u - z^2)$. Therefore, the equations (1.21) and (1.22) give

$$(v')^2 = \epsilon(2v^2+z)v' - v^4 - zv^2 + \tfrac{1}{4}(u - z^2), \tag{3.10}$$

and

$$v' = \frac{\epsilon}{(2\alpha-\epsilon)}\left[(2\alpha-\epsilon)v^2 + uv + \frac{\epsilon}{4}u' + \frac{1}{2}(2\alpha-\epsilon)z\right], \tag{3.11}$$

respectively, and the quadratic equation in v is

$$(4uv + \epsilon u')^2 = 4(2\alpha-\epsilon)^2 u. \tag{3.12}$$

The equations (3.10) and (3.12) give one-to-one correspondence between solutions $v(z)$ of PII and solutions $u(z)$ of the following second-order second-degree Painlevé-type equation:

$$[4uu'' - 3(u')^2 + 8zu^2 + 4(2\alpha-\epsilon)^2 u]^2 = 64u^5. \tag{3.13}$$

IV. PAINLEVÉ III

Let $v(z)$ be a solution of PIII equation

$$v'' = \frac{1}{v}(v')^2 - \frac{1}{z}v' + \gamma v^3 + \frac{1}{z}(\alpha v^2 + \beta) + \frac{\delta}{v}. \tag{4.1}$$

Then, for PIII, the equation (1.22) takes the following form:

$$(\phi_4 v^4 + \phi_3 v^3 + \phi_2 v^2 + \phi_1 v + \phi_0)v' + \psi_6 v^6 + \psi_5 v^5 + \psi_4 v^4 + \psi_3 v^3 + \psi_2 v^2 + \psi_1 v + \psi_0 = 0, \tag{4.2}$$

where

$$\phi_4 = 2\gamma - 2B_4 - A_2^2, \quad \phi_3 = \frac{2\alpha}{z} - B_3 - A_1A_2 - A_2' - \frac{1}{z}A_2, \quad \phi_2 = -\left(A_1' + \frac{1}{z}A_1\right),$$

$$\phi_1 = \frac{2\beta}{z} + B_1 + A_0A_1 - A_0' - \frac{1}{z}A_0, \quad \phi_0 = A_0^2 + 2B_0 + 2\delta,$$

$$\psi_6 = -A_2(B_4+\gamma), \quad \psi_5 = -\left(B_4' + \frac{2}{z}B_4 + A_2B_3 + \gamma A_1 + \frac{\alpha}{z}A_2\right),$$

$$\psi_4 = A_0B_4 - B_3' - \frac{2}{z}B_3 - A_2B_2 - \gamma A_0 - \frac{\alpha}{z}A_1, \tag{4.3}$$

$$\psi_3=A_0B_3-B_2'-\frac{2}{z}B_2-A_2B_1-\frac{\beta}{z}A_2-\frac{\alpha}{z}A_0,$$

$$\psi_2=A_0B_2-B_1'-\frac{2}{z}B_1-A_2B_0-\frac{\beta}{z}A_1-\delta A_2,$$

$$\psi_1=A_0B_1-B_0'-\frac{2}{z}B_0-\frac{\beta}{z}A_0-\delta A_1,\quad \psi_0=A_0(B_0-\delta).$$

As an example, let $A_0=0$, $A_1=2/z$, $B_1=-2\beta/z$, and $B_0=-\delta$. Then one gets $\phi_0=\phi_1=\phi_2=0$ and $\psi_0=\psi_1=\psi_2=0$. Moreover, if $\phi_4=\psi_6=0$, then either $A_2=0$, $B_4=\gamma$ or $A_2=2\sqrt{\gamma}$, $B_4=-\gamma$, where γ can be taken with either sign.

Case i: If $A_2=0$, $B_4=\gamma$, then equation (1.26) takes the following form:

$$(\psi_5v^2+\psi_4v+\psi_3)^2+\frac{2}{z}\phi_3v(\psi_5v^2+\psi_4v+\psi_3)-\phi_3^2\left(\gamma v^4+B_3v^3+B_2v^2-\frac{2\beta}{z}v-\delta\right)=0. \tag{4.4}$$

To reduce the equation (4.4) to a quadratic equation in v one may set the coefficients of v^4 and v^3 to zero. Then, one obtains $B_3=(2/z)(\alpha+2\sqrt{\gamma})$, and without loss of generality, one may take $B_2=u$. With these choices the quadratic equation in v takes the following form,

$$\begin{aligned}8[\gamma z^3u'+2(\alpha+\sqrt{\gamma})(\alpha+3\sqrt{\gamma})]v^2+8[(\alpha+2\sqrt{\gamma})(zu'+u)+4\gamma\beta]v\\ +z^2(zu'+2u)^2+16\gamma\delta z^2=0,\end{aligned} \tag{4.5}$$

and the transformations (1.21) and (1.22) become

$$(v')^2=\frac{2}{z}vv'+\gamma v^4+\frac{2}{z}(\alpha+2\sqrt{\gamma})v^3+uv^2-\frac{2\beta}{z}v-\delta \tag{4.6}$$

and

$$v'=\frac{-1}{4z\sqrt{\gamma}}[4\gamma zv^2+4(\alpha+\sqrt{\gamma})v+z^2u'+2zu], \tag{4.7}$$

respectively. Then, the transformations (4.5) and (4.6) give one-to-one correspondence between solutions $v(z)$ of PIII and solutions $y(x)$ of the following second-order second-degree Painlevé-type equation

$$x^2(\ddot{y})^2=-4(\dot{y})^2(x\dot{y}-y)-\frac{\gamma\delta}{16}(x\dot{y}-y)+\frac{\beta}{16}(\alpha+2\sqrt{\gamma})\dot{y}+\frac{1}{256}[\gamma\beta^2-\delta(\alpha+2\sqrt{\gamma})^2], \tag{4.8}$$

where $y(x)=\frac{1}{16}[z^2u(z)+1]$ and $x=z^2$. The equation (4.8) was first obtained by Cosgrove[17] and labeled as SD-I.b (with $A_1=0$).

Case ii: $A_2=2\sqrt{\gamma}$, $B_4=-\gamma$: The equation (1.26) takes the form of

$$\begin{aligned}(\psi_5v^2+\psi_4v+\psi_3)^2+\frac{2}{z}\phi_3v(\sqrt{\gamma}zv+1)(\psi_5v^2+\psi_4v+\psi_3)\\ +\phi_3^2\left(\gamma v^4-B_3v^3-B_2v^2+\frac{2\beta}{z}v+\delta\right)=0.\end{aligned} \tag{4.9}$$

One may set the coefficients of v^4 and v^3 to zero in order to reduce the equation (4.9) to a quadratic equation in v. Then, one obtains $B_3=-2\sqrt{\gamma}/z$, and without loss of generality one can take $B_2=u$. Then, the equation (1.21) becomes

$$(v')^2=\frac{2}{z}v(\sqrt{\gamma}zv+1)v'-\gamma v^4-\frac{2\sqrt{\gamma}}{z}v^3+uv^2-\frac{2\beta}{z}v-\delta. \tag{4.10}$$

By using the linear transformation $y(x)=z^2u(z)+1$, $2x=z^2$, and $\mu=\alpha-2\sqrt{\gamma}$, the equation (1.22) can be written as

$$v'=\sqrt{\gamma}v^2+\frac{1}{z}\left(\frac{\sqrt{\gamma}}{\mu}y+1\right)v+\frac{1}{2\mu}(\dot{y}-2\beta\sqrt{\gamma}), \tag{4.11}$$

and the quadratic equation for v is

$$4y(\gamma y-\mu^2)v^2+4z[\sqrt{\gamma}y(\dot{y}-2\beta\sqrt{\gamma})+2\beta\mu^2]v+z^2[(\dot{y}-2\beta\sqrt{\gamma})^2+4\delta\mu^2]=0. \tag{4.12}$$

The equations (4.10) and (4.12) give one-to-one correspondence between solutions $v(z)$ of PIII and solutions $y(x)$ of the following equation:

$$x^2[2y^2\ddot{y}-y\dot{y}^2-4(\delta\mu^2-\gamma\beta^2)y-8\beta^2\mu^2]^2=(y^2+4\beta\mu x)^2[y(\dot{y})^2-4(\gamma y-\mu^2)(\delta y+\beta^2)]. \tag{4.13}$$

V. PAINLEVÉ IV

Let $v(z)$ be a solution of PIV

$$v''=\frac{1}{2v}(v')^2+\frac{3}{2}v^3+4zv^2+2(z^2-\alpha)v+\frac{\beta}{v}. \tag{5.1}$$

Then, for PIV the equation (1.22) takes the following form,

$$(\phi_4v^4+\phi_3v^3+\phi_2v^2+\phi_1v+\phi_0)v'+\psi_6v^6+\psi_5v^5+\psi_4v^4+\psi_3v^3+\psi_2v^2+\psi_1v+\psi_0=0, \tag{5.2}$$

where

$$\begin{gathered}
\phi_4=3(1-B_4-\tfrac{1}{2}A_2^2), \quad \phi_3=8z-2B_3-2A_1A_2-A_2', \\
\phi_2=4(z^2-\alpha)-B_2-\tfrac{1}{2}A_1^2-A_0A_2-A_1', \quad \phi_1=-A_0', \quad \phi_0=\tfrac{1}{2}A_0^2+B_0+2\beta, \\
\psi_6=-\tfrac{3}{2}A_2(B_4+1), \quad \psi_5=-(B_4'+\tfrac{1}{2}A_1B_4+\tfrac{3}{2}A_2B_3+4zA_2+\tfrac{3}{2}A_1), \\
\psi_4=\tfrac{1}{2}A_0B_4-B_3'-\tfrac{1}{2}A_1B_3-\tfrac{3}{2}A_2B_2-2(z^2-\alpha)A_2-\tfrac{3}{2}A_0-4zA_1, \\
\psi_3=\tfrac{1}{2}A_0B_3-B_2'-\tfrac{1}{2}A_1B_2-\tfrac{3}{2}A_2B_1-2(z^2-\alpha)A_1-4zA_0, \\
\psi_2=\tfrac{1}{2}A_0B_2-B_1'-\tfrac{1}{2}A_1B_1-\tfrac{3}{2}A_2B_0-\beta A_2-2(z^2-\alpha)A_0, \\
\psi_1=\tfrac{1}{2}A_0B_1-B_0'-\tfrac{1}{2}A_1B_0-\beta A_1, \quad \psi_0=\tfrac{1}{2}A_0(B_0-2\beta).
\end{gathered} \tag{5.3}$$

As an example, let $A_0=0$ and $B_0=-2\beta$. Then one gets $\phi_0=\phi_1=\psi_0=\psi_1=0$. Moreover, setting $\phi_4=\phi_3=\psi_6=\psi_5=0$, one has the following two distinct cases: (i) $A_2=0$, $A_1=0$, $B_4=1$, $B_3=4z$ or (ii) $A_2=2\epsilon$, $A_1=4\epsilon z$, $B_4=-1$, $B_3=-4z$, where $\epsilon=\pm1$.

Case i: In this case Eq. (1.26) takes the form of

$$(\psi_4 v^2+\psi_3 v+\psi_2)^2-\phi_2^2(v^4+4zv^3+B_2v^2+B_1v-2\beta)=0. \tag{5.4}$$

To reduce the equation (5.4) to a quadratic equation in v, one may set the coefficients of v^4 and v^3 to zero. Then, one obtains $B_2=4(z^2-\alpha+\epsilon)$ and, hence, without loss of generality, one can choose $B_1=u$. Then the equations (1.21) and (1.22) become

$$(v')^2=v^4+4zv^3+4(z^2-\alpha+\epsilon)v^2+uv-2\beta \tag{5.5}$$

and

$$v'=\frac{-\epsilon}{4}(4v^2+8zv+u'), \tag{5.6}$$

respectively. The equations (5.5) and

$$8[u'+8(\alpha-\epsilon)]v^2+16(zu'-u)v+(u')^2+32\beta=0 \tag{5.7}$$

give one-to-one correspondence between solutions $v(z)$ of PIV and solutions $u(z)$ of the following equation:

$$(u'')^2=4(zu'-u)^2-\tfrac{1}{2}[(u')^2+32\beta](u'+8\alpha-8\epsilon). \tag{5.8}$$

The transformation $u=8(y-\mu z)$, where $\mu=\frac{1}{3}(\alpha-\epsilon)$, transforms the equation (5.8) to the following equation,

$$(y'')^2=-4(y')^3+4(zy'-y)^2+2(6\mu^2-\beta)y'-4\mu(2\mu^2+\beta), \tag{5.9}$$

which was first obtained by Cosgrove[17] and labeled as SD-I.c.

Case ii: In this case Eq. (1.26) can be written as follows:

$$[(\psi_4+\epsilon\phi_2)v^2+(\psi_3+2\epsilon z\phi_2)v+\psi_2]^2=\phi_2^2[(B_2+4z^2)v^2+B_1v-2\beta]. \tag{5.10}$$

It is clear that if one sets $\psi_4+\epsilon\phi_2=0$, then the equation (5.10) reduces to a quadratic equation in v. Thus, one should take $B_2=-4z^2$ and, without loss of generality, $B_1=u$. Then, the equations (1.21) and (1.22) become, respectively,

$$(v')^2=2\epsilon v(v+2z)v'-v^4-4zv^3-4z^2v^2+uv-2\beta \tag{5.11}$$

and

$$v'=\frac{\epsilon}{12\mu}[12\mu v^2-3(u-8\mu z)v-(\epsilon u'+2zu-4\beta)], \tag{5.12}$$

where $\mu=\frac{1}{3}(\alpha+\epsilon)$. The equations (5.11) and

$$9u^2v^2+2u(3\epsilon u'+6zu-12\beta-72\mu^2)v+(\epsilon u'+2zu-4\beta)^2+288\beta\mu^2=0 \tag{5.13}$$

give one-to-one correspondence between solutions $v(z)$ of PIV and solutions $u(z)$ of the following second-order second-degree Painlevé-type equation:

$$\begin{aligned}&[3uu''-2(u')^2-2\epsilon(zu-2\beta+12\mu^2)u'-2(4z^2-3\epsilon)u^2-8(6\mu^2-\beta)zu+16(6\mu^2-\beta)^2]^2\\&\quad=-27[u^2-16\mu(2\mu^2+\beta)]^2[\epsilon u'+2zu+2\beta-12\mu^2].\end{aligned} \tag{5.14}$$

VI. PAINLEVÉ V

Let $v(z)$ be a solution of PV:

$$v''=\frac{3v-1}{2v(v-1)}(v')^2-\frac{1}{z}v'+\frac{\alpha}{z^2}v(v-1)^2+\frac{\beta(v-1)^2}{z^2v}+\frac{\gamma}{z}v+\frac{\delta v(v+1)}{v-1}. \tag{6.1}$$

Then, for PV, Eq. (1.22) takes the form of

$$\begin{aligned}(\phi_5v^5+\phi_4v^4+\phi_3v^3+\phi_2v^2+\phi_1v+\phi_0)v'+\psi_7v^7+\psi_6v^6\\ +\psi_5v^5+\psi_4v^4+\psi_3v^3+\psi_2v^2+\psi_1v+\psi_0=0,\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
&\phi_5=\frac{2\alpha}{z^2}-B_4-\frac{1}{2}A_2^2,\quad \phi_4=3B_4+\frac{3}{2}A_2^2-\frac{6\alpha}{z^2}-A_2'-\frac{1}{z}A_2,\\
&\phi_3=2B_3+B_2+\frac{1}{2}A_1^2+2A_1A_2+A_0A_2+A_2'+\frac{1}{z}A_2-A_1'-\frac{1}{z}A_1+\frac{2}{z^2}[3\alpha+\beta+\gamma z+\delta z^2],\\
&\phi_2=2B_1+B_2+\frac{1}{2}A_1^2+2A_0A_1+A_0A_2+A_1'+\frac{1}{z}A_1-A_0'-\frac{1}{z}A_0-\frac{2}{z^2}[\alpha+3\beta+\gamma z-\delta z^2],\\
&\phi_1=3B_0+\frac{3}{2}A_0^2+\frac{6\beta}{z^2}+A_0'+\frac{1}{z}A_0,\quad \phi_0=-\left(\frac{2\beta}{z^2}+B_0+\frac{1}{2}A_0^2\right),\\
&\psi_7=-\frac{1}{2}A_2\left(B_4+\frac{2\alpha}{z^2}\right),\quad \psi_6=B_4\left(\frac{3}{2}A_2+\frac{1}{2}A_1-\frac{2}{z}\right)-\frac{1}{2}A_2B_3+\frac{\alpha}{z^2}(3A_2-A_1)-B_4',\\
&\psi_5=B_4\left(\frac{1}{2}A_1+\frac{3}{2}A_0+\frac{2}{z}\right)+B_3\left(\frac{3}{2}A_2+\frac{1}{2}A_1-\frac{2}{z}\right)-\frac{1}{2}A_2B_2\\
&\qquad -\frac{A_2}{z^2}(3\alpha+\beta+\gamma z+\delta z^2)+\frac{\alpha}{z^2}(3A_1-A_0)+B_4'-B_3',\\
&\psi_4=B_3\left(\frac{1}{2}A_1+\frac{3}{2}A_0+\frac{2}{z}\right)+B_2\left(\frac{3}{2}A_2+\frac{1}{2}A_1-\frac{2}{z}\right)-\frac{1}{2}A_2B_1-\frac{1}{2}A_0B_4\\
&\qquad -\frac{A_1}{z^2}(3\alpha+\beta+\gamma z+\delta z^2)+\frac{A_2}{z^2}(\alpha+3\beta+\gamma z-\delta z^2)+\frac{3\alpha}{z^2}A_0+B_3'-B_2',\\
&\psi_3=B_2\left(\frac{1}{2}A_1+\frac{3}{2}A_0+\frac{2}{z}\right)+B_1\left(\frac{3}{2}A_2+\frac{1}{2}A_1-\frac{2}{z}\right)-\frac{1}{2}A_2B_0-\frac{1}{2}A_0B_3\\
&\qquad -\frac{A_0}{z^2}(3\alpha+\beta+\gamma z+\delta z^2)+\frac{A_1}{z^2}(\alpha+3\beta+\gamma z-\delta z^2)-\frac{3\beta}{z^2}A_2+B_2'-B_1',\\
&\psi_2=B_1\left(\frac{1}{2}A_1+\frac{3}{2}A_0+\frac{2}{z}\right)+B_0\left(\frac{3}{2}A_2+\frac{1}{2}A_1-\frac{2}{z}\right)\\
&\qquad +\frac{A_0}{z^2}(\alpha+3\beta+\gamma z-\delta z^2)+\frac{\beta}{z^2}(A_2-3A_1)+B_1'-B_0',
\end{aligned} \tag{6.3}$$

$$\psi_1=B_0\left(\frac{3}{2}A_0+\frac{1}{2}A_1+\frac{2}{z}\right)-\frac{1}{2}A_0B_1+\frac{\beta}{z^2}(A_1-3A_0)+B_0', \quad \psi_0=\frac{-1}{2}A_0\left(B_0-\frac{2\beta}{z^2}\right).$$

As an example, let

$$A_1=\frac{-2}{z}(zA_0+1), \quad A_2=\frac{1}{z}(zA_0+2),$$
$$B_3=-\left(2B_2+3B_1+4B_0-\frac{2\gamma}{z}+4\delta\right), \quad B_4=B_2+2B_1+3B_0-\frac{2\gamma}{z}+2\delta, \tag{6.4}$$

and let $\phi_0=\phi_1=\psi_0=\psi_1=0$. Then, Eq. (7.2) can be written as

$$\phi_5v'+\psi_7v^2+(\psi_6+3\psi_7)v-\psi_2=0, \tag{6.5}$$

and the equation (1.26) can be written as

$$[\psi_7v^2+(\psi_6+3\psi_7v-\psi_2+\tfrac{1}{2}\phi_5(A_2v^2+A_1v+A_0)]^2$$
$$=\phi_5^2[(B_4+\tfrac{1}{4}A_2^2)v^4+(B_3+\tfrac{1}{2}A_1A_2)v^3+(\tfrac{1}{4}B_2A_1^2+\tfrac{1}{2}A_0A_2)v^2$$
$$+(B_1+\tfrac{1}{2}A_0A_1)v+(B_0+\tfrac{1}{4}A_0^2)]. \tag{6.6}$$

Here $\psi_0=0$ implies that either $A_0=0$ or $B_0=2\beta/z^2$.

Case i: $A_0=0$: Equations (6.4) and $\phi_0=\phi_1=\psi_1=0$ imply that $A_1=-2/z$, $A_2=2/z$, and $B_0=-2\beta/z^2$. If $B_4=(\mu^2-1)/z^2$, where $\mu=1-\sqrt{2\alpha}$ and $\sqrt{2\alpha}$ can take either sign, and without loss of generality $B_1=(1/z^2)(4u+\gamma z-\mu^2+6\beta)$, then Eq. (6.6) reduces to the following quadratic equation for v:

$$Av^2+Bv+C=0, \tag{6.7}$$

where

$$A=8\mu^2[2(zu'+u)+\delta z^2-\mu^2+2\beta]+(4u-\gamma z-3\mu^2+2\beta)^2,$$
$$B=2(4u-\gamma z-3\mu^2+2\beta)[4(zu'-u)+\mu^2-2\beta]-4\mu^2(4u+\gamma z-\mu^2+6\beta), \tag{6.8}$$
$$C=[4(zu'-u)+\mu^2-2\beta]^2+8\beta\mu^2.$$

The equations (1.21) and (1.22) respectively become

$$(v')^2=\frac{1}{z^2}[2zv(v-1)v'+(\mu^2-1)v^4+(4u-\gamma z-3\mu^2+2\beta+2)v^3$$
$$-(8u+2\delta z^2-3\mu^2+6\beta+1)v^2+(4u+\gamma z-\mu^2+6\beta)v-2\beta], \tag{6.9}$$

and

$$v'=\frac{1}{2\mu z}[2\mu\sqrt{2\alpha}v^2-(4u-\gamma z-3\mu^2+2\beta+2\mu)v-(4zu'-4u+\mu^2-2\beta)]. \tag{6.10}$$

The equations (6.9) and (6.7) define one-to-one correspondence between solutions $v(z)$ of PV and solutions $u(z)$ of the following second-order second-degree Painlevé-type equation:

$$z^2(u'')^2 = -4(u')^2(zu'-u) - 2\delta(zu'-u)^2 - [\delta(\mu^2-2\beta) - \tfrac{1}{4}\gamma^2](zu'-u) + \tfrac{1}{2}\gamma(\mu^2+2\beta)u' + \tfrac{1}{8}[\gamma^2(\mu^2-2\beta) - \delta(\mu^2+2\beta)^2]. \tag{6.11}$$

The equation (6.11) was first obtained by Cosgrov[17] and labeled as SD-I.b.

Case ii: $B_0 = 2\beta/z^2$: Then $A_0 = 2(\mu-1)/z$, $B_1 = -\frac{1}{2}A_0A_1$, where $(\mu-1)^2 = -2\beta$. With out loss of generality, let $B_2 = (1/z^2)[u - 6(\mu-1)^2 - 6(\mu-1) - 1 + 2\gamma z - 2\delta z^2]$. Then Eq. (6.4) implies that $B_4 = (1/z^2)(u-\mu^2)$, $B_3 = (-2/z^2)[u + \gamma z - \mu(2\mu-1)]$. With these choices, the equation (6.6) becomes

$$Av^2 + Bv + C = 0, \tag{6.12}$$

where

$$\begin{aligned} A &= u[u^2 - 2(\mu^2+2\alpha)u + (\mu^2-2\alpha)^2], \\ B &= -4\mu z u u' - 2(u+\gamma z)[u^2 - 2(\mu^2+2\alpha)u + (\mu^2-2\alpha)^2], \\ C &= -[zu' - 2\mu(u+\gamma z)]^2 + (u - 2\delta z^2 + 2\gamma z)(u + \mu^2 - 2\alpha)^2. \end{aligned} \tag{6.13}$$

The equation (1.21) can be written as follows,

$$[zv' - (v-1)(\mu v - \mu - 1)]^2 = uv^4 - 2(u+\gamma z)v^3 + (u - 2\delta z^2 + 2\gamma z)v^2, \tag{6.14}$$

and the equation (1.22) becomes

$$v' = -\frac{1}{z(u+\mu^2-2\alpha)}\{\mu(u-\mu^2+2\alpha)v^2 + [zu' - u - 2\gamma\mu z + (2\mu-1)(\mu^2-2\alpha)]v - (\mu-1)(u+\mu^2-2\alpha)\}. \tag{6.15}$$

Let $u(z)$ be a solution of the following second-order second-degree equation of Painlevé type:

$$\begin{aligned} &[2uu'' - (u')^2 + 2\delta u^2 + 2\gamma u - 2\delta(\mu^2-2\alpha)^2 - 2\gamma^2(\mu^2+2\alpha)]^2 \\ &= 8[u^2 - \gamma z(\mu^2-2\alpha)^2]^2\{u(u')^2 + (2\delta u + \gamma^2) \\ &\quad \times [u^2 - 2(\mu^2+2\alpha)u + (\mu^2-2\alpha)^2]\}. \end{aligned} \tag{6.16}$$

Then Eqs. (6.12) and (6.14) give one-to-one correspondence between solutions $v(z)$ of PV and $u(z)$ of the equation (6.16).

VII. PAINLEVÉ VI

Let $v(z)$ be a solution of PVI

$$v'' = \frac{1}{2}\left(\frac{1}{v} + \frac{1}{v-1} + \frac{1}{v-z}\right)(v')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{v-z}\right)v' + \frac{v(v-1)(v-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{v^2} + \frac{\gamma(z-1)}{(v-1)^2} + \frac{\delta z(z-1)}{(v-z)^2}\right). \tag{7.1}$$

Then, for PVI, Eq. (1.22) takes the form of

$$(\phi_6 v^6 + \phi_5 v^5 + \phi_4 v^4 + \phi_3 v^3 + \phi_2 v^2 + \phi_1 v + \phi_0)v' + \psi_8 v^8 + \psi_7 v^7 + \psi_6 v^6 + \psi_5 v^5 + \psi_4 v^4 + \psi_3 v^3 + \psi_2 v^2 + \psi_1 v + \psi_0 = 0, \tag{7.2}$$

where

$$\phi_6=\frac{2\alpha}{z^2(z-1)^2}-B_4-\frac{1}{2}A_2^2,\quad \phi_5=2(z+1)B_4+(z+1)A_2^2-\frac{4\alpha(z+1)}{z^2(z-1)^2}-A_2'-\frac{(2z-1)}{z(z-1)}A_2,$$

$$\phi_4=\frac{z}{(z-1)}A_2-\frac{(2z-1)}{z(z-1)}(A_1-A_2)+\frac{1}{2}A_1^2+A_0A_2+(z+1)A_1A_2-\frac{3}{2}zA_2^2+B_2+(z+1)B_3$$
$$-3zB_4+(z+1)A_2'-A_1'+\frac{2}{z^2(z-1)^2}[\alpha(z^2+4z+1)+\beta z+(\delta z+\gamma)(z-1)],$$

$$\phi_3=\frac{z}{(z-1)}(A_1-A_2)-\frac{(2z-1)}{z(z-1)}(A_0-A_1)+2A_0A_1-2zA_1A_2+2B_1-2zB_3$$
$$+(z+1)A_1'-A_0'-zA_2'-\frac{4}{z(z-1)^2}[(\alpha+\beta)(z+1)+(\gamma+\delta)(z-1)],$$

$$\phi_2=\frac{z}{(z-1)}(A_0-A_1)+\frac{(2z-1)}{z(z-1)}A_0+\frac{3}{2}A_0^2-(z+1)A_0A_1-zA_0A_2-\frac{1}{2}zA_1^2+3B_0-(z+1)B_1$$
$$-zB_2-zA_1'+(z+1)A_0'-\frac{2}{z(z-1)^2}[\alpha z+\beta(z^2+4z+1)+(\gamma z+\delta)(z-1)],$$

$$\phi_1=-\left[2(z+1)B_0+(z+1)A_0^2+\frac{4\beta(z+1)}{(z-1)^2}+zA_0'+\frac{z}{(z-1)}A_0\right],$$

$$\phi_0=z\left[B_0+\frac{1}{2}A_0^2+\frac{2\beta}{(z-1)^2}\right],$$

$$\psi_8=-\frac{1}{2}A_2\left[B_4+\frac{2\alpha}{z^2(z-1)^2}\right],$$

$$\psi_7=B_4\left[(z+1)A_2+\frac{1}{2}A_1-\frac{(2z-1)}{z(z-1)}\right]-\frac{1}{2}A_2B_3+\frac{\alpha}{z^2(z-1)^2}[2(z+1)A_2-A_1]-B_4', \tag{7.3}$$

$$\psi_6=B_4\left[\frac{3}{2}(A_0-zA_2)+\frac{2z}{(z-1)}+\frac{2(2z-1)}{z(z-1)}\right]+B_3\left[(z+1)A_2+\frac{1}{2}A_1-\frac{2(2z-1)}{z(z-1)}\right]$$
$$-\frac{1}{2}A_2B_2+(z+1)B_4'-B_3'+\frac{\alpha}{z^2(z-1)^2}[2(z+1)A_1-A_0]$$
$$-\frac{A_2}{z^2(z-1)^2}[\alpha(z^2+4z+1)+\beta z+(\delta z+\gamma)(z-1)],$$

$$\psi_5=B_3\left[\frac{3}{2}(A_0-zA_2)+\frac{2z}{(z-1)}+\frac{2(2z-1)}{z(z-1)}\right]+B_2\left[(z+1)A_2+\frac{1}{2}A_1-\frac{2(2z-1)}{z(z-1)}\right]-\frac{1}{2}A_2B_1$$
$$-B_4\left[\frac{1}{2}zA_1+(z+1)A_0+\frac{2z}{(z-1)}\right]+\frac{2\alpha(z+1)}{z^2(z-1)^2}A_0-\frac{A_1}{z^2(z-1)^2}[\alpha(z^2+4z+1)+\beta z$$
$$+(\delta z+\gamma)(z-1)]+\frac{2A_2}{z(z-1)^2}[(\alpha+\beta)(z+1)+(\gamma+\delta)(z-1)]+(z+1)B_3'-zB_4'-B_2',$$

$$\psi_4=B_2\left[\frac{3}{2}(A_0-zA_2)+\frac{2z}{(z-1)}+\frac{2(2z-1)}{z(z-1)}\right]+B_1\left[(z+1)A_2+\frac{1}{2}A_1-\frac{2(2z-1)}{z(z-1)}\right]-\frac{1}{2}A_2B_0$$
$$-B_3\left[\frac{1}{2}zA_1+(z+1)A_0+\frac{2z}{(z-1)}\right]+\frac{1}{2}zA_0B_4-\frac{A_0}{z^2(z-1)^2}[\alpha(z^2+4z+1)+\beta z$$
$$+(\delta z+\gamma)(z-1)]+\frac{2A_1}{z(z-1)^2}[(\alpha+\beta)(z+1)+(\gamma+\delta)(z-1)]$$
$$-\frac{A_2}{z(z-1)}[\alpha z+\beta(z^2+4z+1)+\gamma z(z-1)+\delta(z-1)]+(z+1)B_2'-zB_3'-B_1',$$

$$\psi_3=B_1\left[\frac{3}{2}(A_0-zA_2)+\frac{2z}{(z-1)}+\frac{2(2z-1)}{z(z-1)}\right]+B_0\left[(z+1)A_2+\frac{1}{2}A_1-\frac{2(2z-1)}{z(z-1)}\right]+\frac{1}{2}zA_0B_3$$
$$-B_2\left[\frac{1}{2}zA_1+(z+1)A_0+\frac{2z}{(z-1)}\right]+\frac{2\beta(z+1)}{(z-1)^2}A_2+\frac{2A_0}{z(z-1)^2}[(\alpha+\beta)(z+1)$$
$$+(\gamma+\delta)(z-1)]-\frac{A_1}{z(z-1)}[\alpha z+\beta(z^2+4z+1)+(\gamma z+\delta)(z-1)]+(z+1)B_1'-zB_2'-B_0',$$

$$\psi_2=B_0\left[\frac{3}{2}(A_0-zA_2)+\frac{2z}{(z-1)}+\frac{2(2z-1)}{z(z-1)}\right]-B_1\left[\frac{1}{2}zA_1+(z+1)A_0+\frac{2z}{(z-1)}\right]$$
$$+\frac{1}{2}zA_0B_2+(z+1)B_0'-zB_1'+\frac{\beta}{(z-1)^2}[2(z+1)A_1-zA_2]$$
$$-\frac{A_0}{z(z-1)}[\alpha z+\beta(z^2+4z+1)+(\gamma z+\delta)(z-1)],$$

$$\psi_1=\frac{\beta}{(z-1)^2}[2(z+1)A_0-zA_1]+\frac{1}{2}zA_0B_1-B_0\left[(z+1)A_0+\frac{1}{2}zA_1+\frac{2z}{(z-1)}\right]-zB_0',$$

$$\psi_0=\frac{z}{2}A_0\left[B_0-\frac{2\beta}{(z-1)^2}\right].$$

As an example, let

$$A_1=\frac{-1}{z(z-1)}[(z^2-1)A_0+2],\quad A_2=\frac{1}{z(z-1)}[(z-1)A_0+2],$$

$$B_3=\frac{-1}{z^3(z-1)}[z^2(z^2-1)B_2+z(z-1)(z^2+z+1)B_1+(z-1)(z^3+z^2+z+1)B_0-2\gamma z^2-2\delta],\tag{7.4}$$

$$B_4=\frac{1}{z^3(z-1)}[z^2(z-1)B_2+z(z^2-1)B_1+(z-1)(z^2+z+1)B_0-2\gamma z-2\delta],$$

and $\phi_0=\phi_1=\psi_0=\psi_1=0$. Then, the equation (7.2) takes the following form,

$$\phi_6v'+\psi_8v^2+[\psi_7+(z+1)\psi_8]v+\frac{1}{z^2}\psi_2=0,\tag{7.5}$$

and the equation (1.26) can be written as

$$\left(\psi_8 v^2+[\psi_7+(z+1)\psi_8]v+\frac{1}{z^2}\psi_2+\frac{1}{2}\phi_6(A_2v^2+A_1v+A_0)\right)^2$$
$$=\phi_6^2[(B_4+\tfrac{1}{4}A_2^2)v^4+(B_3+\tfrac{1}{2}A_1A_2)v^3+(B_2+\tfrac{1}{4}A_1^2+\tfrac{1}{2}A_0A_2)v^2$$
$$+(B_1+\tfrac{1}{2}A_0A_1)v+(B_0+\tfrac{1}{4}A_0^2)]. \tag{7.6}$$

The equation $\psi_0=0$ implies that either $A_0=0$ or $B_0=2\beta/(z-1)^2$.

Case i: $A_0=0$: Then, the equation $\phi_0=0$ implies that $B_0=-2\beta/(z-1)^2$ and then the equations $\phi_1=\psi_1=0$ are satisfied identically. Let $B_4=(\mu^2-1)/z^2(z-1)^2$, where $\mu=1-\sqrt{2\alpha}$ and $\sqrt{2\alpha}$ can take either sign, and without loss of generality, let $B_2=-[1/z^2(z-1)^2][4(z+1)u+(\beta-\alpha+\sqrt{\alpha})(3z+1)+(\gamma-\delta)(3z-1)]$. Then the equation (7.6) reduces to the following quadratic equation for v:

$$Av^2+Bv+C=0,$$
$$A=4\mu^2[4z(z-1)u'+4u+2\nu z-\kappa]+[4u-2\lambda(z-1)+\nu-\mu^2]^2, \tag{7.7}$$
$$B=2z[4u-2\lambda(z-1)+\nu-\mu^2][4(z-1)u'-4u-\nu]-4\mu^2z[4u+2(\gamma+\beta)(z-1)+\nu+4\beta],$$
$$C=z^2[4(z-1)u'-4u-\nu]^2+8\beta\mu^2z^2,$$

where $\kappa=\alpha-\beta+\gamma-\delta-\sqrt{2\alpha}+1$, $\lambda=\alpha+\delta-\sqrt{2\alpha}$, and $\nu=\beta+\gamma-\alpha-\delta+\sqrt{2\alpha}$. The equation (1.21) can be written as

$$[z(z-1)v'-v(v-1)]^2=\mu^2v^4+[4u-2\lambda(z-1)+\nu-\mu^2]v^3-[4(z+1)u+3\nu z-\kappa]v^2$$
$$+z[4u+2(\gamma+\beta)(z-1)+\nu+4\beta]v-2\beta z^2, \tag{7.8}$$

and the equation (1.22) becomes

$$v'=\frac{1}{2\mu z(z-1)}\{2\mu\sqrt{2\alpha}v^2-[4u-2\lambda(z-1)+\nu-\mu^2+2\mu]v-z[4(z-1)u'-4u-\nu]\}. \tag{7.9}$$

Equations (7.7) and (7.8) give one-to-one correspondence between solutions $v(z)$ of PVI and solutions $u(z)$ of the following second-order second degree equation of Painlevé type:

$$z^2(z-1)^2(u'')^2=-4u'(zu'-u)^2+4(u')^2(zu'-u)+\kappa(u')^2+\lambda(\gamma+\beta)(zu'-u)$$
$$+\tfrac{1}{4}[4(\gamma-\beta)(\mu^2-\lambda)+\nu^2]u'+\tfrac{1}{4}[\lambda^2(\gamma-\beta)+(\gamma+\beta)^2(\mu^2-\lambda)]. \tag{7.10}$$

The equation (7.10) was first obtained by Cosgrove[17] and labeled as SD-I.a.

Case ii: $B_0=2\beta/(z-1)^2$: Then $A_0=2(\mu-1)/(z-1)$, $B_0=\frac{1}{4}A_0^2$, and $B_1=-\frac{1}{2}A_0A_1$, where $(\mu-1)^2=-2\beta$. Without loss of generality, let

$$B_2=\frac{1}{z^2(z-1)^2}[zu-\mu^2(z^2+4z+1)+2\mu z(z+2)-(z^2+z-1)+2\gamma z(z-1)+2\delta(z-1)]. \tag{7.11}$$

Then one obtains $B_4=[1/z^2(z-1)^2](u-\mu^2)$ and $B_3=[-1/z^2(z-1)^2][(z+1)(u-2\mu^2)+2\mu z+\lambda(z-1)]$, where $\lambda=2\gamma+2\delta-1$. With these choices the equation (7.6) yields the following quadratic equation for v:

$$Av^2+Bv+C=0, \tag{7.12}$$

where

$$\begin{aligned} A&=u[u^2-2(\mu^2+2\alpha)u+(\mu^2-2\alpha)^2],\\ B&=-(4\mu z(z-1)uu'+[(z+1)u+\lambda(z-1)][u^2-2(\mu^2+2\alpha)u+(\mu^2-2\alpha)^2]),\\ C&=-(z(z-1)u'-\mu[(z+1)u+\lambda(z-1)])^2+[zu+2\gamma(z-1)^2+\lambda(z-1)](u+\mu^2-2\alpha)^2. \end{aligned}\tag{7.13}$$

The equations (1.21) and (1.22) become

$$\begin{aligned} &[z(z-1)v'-\mu v^2+(\mu z-z+\mu)v-(\mu-1)z]^2\\ &\quad=uv^4-[(z+1)u+\lambda(z-1)]v^3+[zu+2\gamma(z-1)^2+\lambda(z-1)]v^2, \end{aligned}\tag{7.14}$$

and

$$\begin{aligned} v'=\frac{-1}{z(z-1)(u+\mu^2-2\alpha)}&\{\mu(u-\mu^2+2\alpha)v^2+[z(z-1)u'-z(u+\mu^2-2\alpha)\\ &-\mu\lambda(z-1)+\mu(\mu^2-2\alpha)(z+1)]v-(\mu-1)z(u+\mu^2-2\alpha)\}, \end{aligned}\tag{7.15}$$

respectively. Let $u(z)$ be a solution of the following second-order second-degree equation of Painlevé type:

$$\begin{aligned} &[4z^2u^2u''-2z^2u(u')^2+4zu^2u'+P_4(u)]^2\\ &\quad=\left[\frac{(z+1)}{(z-1)}u^2-\lambda(\mu^2-2\alpha)\right]^2[4z^2u(u')^2+Q_4(u)],\\ &P_4(u):=u^4+(\lambda-4\gamma-\mu^2-2\alpha)u^3+[\lambda^2(\mu^2+2\alpha)+(\lambda-4\gamma)(\mu^2-2\alpha)^2]u-\lambda^2(\mu^2-2\alpha)^2,\\ &Q_4(u):=[u^2+2(\lambda-4\gamma)u+\lambda^2][u^2-2(\mu^2+2\alpha)u+(\mu^2-2\alpha)^2]. \end{aligned}\tag{7.16}$$

Then, the equations (7.12) and (7.14) gives one-to-one correspondence between solutions $v(z)$ of PVI and $u(z)$ of the equation (7.16).

[1] P. Painlevé, Bull. Soc. Math. Fr. **28**, 214 (1900); Acta. Math. **25**, 1 (1912).
[2] B. Gambier, Acta Math. **33**, 1 (1909).
[3] R. Fuchs, Math. Ann. **63**, 301 (1907).
[4] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[5] N. P. Erugin, Dokl. Akad. Nauk BSSR **2** (1958).
[6] A. S. Fokas and M. J. Ablowitz, J. Math. Phys. **23**, 2033 (1982).
[7] U. Muğan and A. S. Fokas, J. Math. Phys. **33**, 2031 (1992).
[8] U. Muğan, A. Sakka, J. Math. Phys. **36**, 1284 (1995).
[9] A. S. Fokas and X. Zhou, Commun. Math. Phys. **144**, 601 (1992).
[10] A. S. Fokas, U. Muğan, and X. Zhou, Inverse Probl. **8**, 757 (1992).
[11] U. Muğan and A. Sakka, J. Phys. A **28**, 4109 (1995).
[12] C. M. Cosgrove, Stud. Appl. Math. **90**, 119 (1993).
[13] J. Chazy, Acta Math. **34**, 317 (1911).
[14] R. Garnier, Ann. Sci. Ec. Norm. Super., **29**, 1 (1912).
[15] F. Bureau, Ann. Math. **66**, 1 (1964).
[16] F. Bureau, Ann. Math. **91**, 163 (1972).
[17] C. M. Cosgrove and G. Scoufis, Stud. Appl. Math. **88**, 25 (1993).
[18] A. S. Fokas and Y. C. Yortsos, Lett. Nuovo Cimento **30**, 539 (1981).
[19] C. M. Cosgrove, J. Phys. A **10**, 1481 (1977).
[20] C. M. Cosgrove, J. Phys. A **10**, 2093 (1977).
[21] C. M. Cosgrove, J. Phys. A **11**, 2405 (1978).

[22] R. Chalkley, J. Diff. Eqns. **68**, 72 (1987).
[23] P. Appell, J. Math. Pures Appl. **5**, 361 (1889).
[24] A. Sakka and U. Muğan, J. Phys. A **30**, 5159 (1997).
[25] A. Sakka and U. Muğan, J. Phys. A **31**, 2471 (1998).